JOURNAL OF APPROXIMATION THEORY **11**, 221–224 (1974)

Comonotone Polynomial Approximation

ELI PASSOW AND LOUIS RAYMON

Department of Mathematics, Temple University, Philadelphia, Pennsylvania 19122

AND

JOHN A. ROULIER

Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27607

Communicated by Oved Shisha

Let P_n be the set of all algebraic polynomials of degree n or less. For $f \in C[a, b]$, the degree of approximation to f by polynomials in P_n is $E_n(f) = \inf\{\| f - p \| : p \in P_n\}$, where the norm is the uniform norm. Jackson's theorem [1] states that there exists $C > 0$ such that $E_n(f) \leqslant C\omega(f; 1/n)$, where $\omega(f; \delta)$ is the modulus of continuity of f.

f is said to be piecewise monotone if it has only a finite number of local maxima and minima in $[a, b]$. The local maxima and minima in (a, b) are called the peaks of f. Wolibner [5] has shown that for any $\epsilon > 0$ there exists a polynomial, p, such that $\| f - p \| < \epsilon$ and p is comonotone with f; i.e., p increases and decreases simultaneously with f. Let $E_n^*(f) = \inf\{\| f - p \| : p \in P_n, p \text{ comonotone with } f\}$. Clearly $E_n^*(f) \geqslant E_n(f)$. We seek an upper bound on $E_n^*(f)$. For f monotone, Lorentz and Zeller [3] have shown that there exists $C_1 > 0$ such that $E_n^*(f) \leqslant C_1\omega(f; 1/n)$. Newman, Passow, and Raymon [4] have obtained results of a modified nature. They have shown that there exists $p \in P_n$ satisfying $\| f - p \| < C_2\omega(f; 1/n)$, C_2 an absolute constant, such that f and p are comonotone except in certain neighborhoods (whose diameters tend to zero with n) of the peaks. In this note we obtain a comonotone approximation on the entire interval $[a, b]$, but at a sacrifice in the accuracy of approximation.

LEMMA 1. *Let $f \in C^{(j+1)}[a, b]$ and suppose that $f(u) = 0$, $u \in (a, b)$. Let $g(x) = f[x, u]$, the divided difference of f, where we define $f[u, u] = f'(u)$. Then $g \in C^j[a, b]$ and $\| g^{(j)} \| \leqslant (j + 1)^{-1} \| f^{(j+1)} \|$.*

Proof. $g(x) = f[x, u] = \int_0^1 f'((x-u)t+u)\,dt$, [2, p. 250]. Thus,

$$g'(x) = \int_0^1 t f''((x-u)t+u)\,dt,$$
$$\vdots$$
$$g^{(j)}(x) = \int_0^1 t^j f^{(j+1)}((x-u)t+u)\,dt.$$

Therefore $g \in C^j[a, b]$ and $\| g^{(j)} \| \leqslant \| f^{(j+1)} \| \int_0^1 t^j\,dt = (j+1)^{-1} \| f^{(j+1)} \|$.

LEMMA 2. *Let f be a piecewise monotone function, with peaks at $x_1, x_2, \ldots, x_k$, and suppose that $f \in C^{(j+k+1)}[a, b]$. Let*

$$g(x) = \sum_{i=1}^{k} \Bigg[\prod_{\substack{l=1 \\ l \neq i}}^{k} (x_i - x_l) \Bigg]^{-1} f'[x, x_i], \qquad \text{where} \quad f'[x_i, x_i] = f''(x_i).$$

Then

(a) $g \in C^{(j+k-1)}[a, b]$;

(b) $\displaystyle \| g^{(j+k-1)} \| \leqslant \frac{\| f^{(j+k+1)} \|}{(j+k)} \sum_{i=1}^{k} \Bigg[\prod_{\substack{l=1 \\ l \neq i}}^{k} | x_i - x_l | \Bigg]^{-1}$;

(c) $\displaystyle \| g^{(j)} \| \leqslant \frac{j!}{(j+k)!} \| f^{(j+k+1)} \|$.

Proof. Since $f' \in C^{(j+k)}[a, b]$, by Lemma 1, $f'[x, x_i] \in C^{(j+k-1)}[a, b]$, so that $g \in C^{(j+k-1)}[a, b]$, proving (a).

Now

$$g(x) = \sum_{i=1}^{k} \Bigg[\prod_{\substack{l=1 \\ l \neq i}}^{k} (x_i - x_l) \Bigg]^{-1} f'[x, x_i]$$
$$= \sum_{i=1}^{k} \Bigg[\prod_{\substack{l=1 \\ l \neq i}}^{k} (x_i - x_l) \Bigg]^{-1} \int_0^1 f''((x - x_i)t + x_i)\,dt.$$

Therefore,

$$g^{(j+k-1)}(x) = \sum_{i=1}^{k} \Bigg[\prod_{\substack{l=1 \\ l \neq i}}^{k} (x_i - x_l) \Bigg]^{-1} \int_0^1 t^{(j+k-1)} f^{(j+k+1)}((x - x_i)t + x_i)\,dt,$$

and (b) follows from this.

To prove (c), let $g_1(x) = f'[x, x_1]$ and $g_{i+1}(x) = g_i[x, x_{i+1}]$, $i = 1, 2,..., k-1$. Then $g_{i+1}(x) = f'[x, x_1 ,..., x_{i+1}]$, $i = 1, 2,..., k - 1$, [2, p. 248]. Hence,

$$g_k(x) = f'[x, x_1 ,..., x_k] = \sum_{i=1}^{k} \left[\prod_{\substack{l=1 \\ l \neq i}}^{k} (x_i - x_l)\right]^{-1} f'[x, x_i] \tag{1}$$

the last expression being equal to $g(x)$ [2, p. 255]. By Lemma 1, $g_1 \in C^{(j+k-1)}[a, b]$ and $\| g_1^{(j+k-1)} \| \leqslant (j + k)^{-1} \| f^{(j+k+1)} \|$, $g_2 \in C^{(j+k-2)}[a, b]$ and $\| g_2^{(j+k-2)} \| \leqslant (j + k - 1)^{-1} \| g_1^{(j+k-1)} \|,..., g_k \in C^j[a, b]$ and $\| g_k^{(j)} \| \leqslant (j + 1)^{-1} \| g_{k-1}^{(j+1)} \|$.
Thus

$$\| g^{(j)} \| = \| g_k^{(j)} \| \leqslant \left[\prod_{l=1}^{k} (j + l)\right]^{-1} \| f^{(j+k+1)} \| = \frac{j!}{(j + k)!} \| f^{(j+k+1)} \|,$$

and the proof of the lemma is complete.

THEOREM 1. *Let f be a piecewise monotone function with peaks at $x_1 , x_2 ,..., x_k$, and suppose that $f \in C^{(j+k+1)}[a, b]$. Then there exists d_j such that, for $n > 2(k + j)$,*

$$E_n^*(f) \leqslant \frac{d_j(b - a)^{k+1} \| f^{(j+k+1)} \|}{n^j}.$$

Proof. Define g as in Lemma 2, and note from (1) that

$$g(x) = \frac{f'(x)}{\prod_{i=1}^{k} (x - x_i)},$$

since $f'(x_i) = 0$ for $i = 1, 2,..., k$. Thus g maintains a constant sign on $[a, b]$, which, we may assume, is nonnegative. Therefore, there exists $q \in P_{n-k-1}$ such that $q(x) \geqslant 0$ on $[a, b]$ and $\| g - q \| \leqslant 2E_{n-k-1}(g)$. Hence,

$$\left| \frac{f'(x)}{\prod_{i=1}^{k} (x - x_i)} - q(x) \right| \leqslant 2E_{n-k-1}(g),$$

so that

$$\left| f'(x) - q(x) \prod_{i=1}^{k} (x - x_i) \right| \leqslant 2(b - a)^k E_{n-k-1}(g).$$

Thus

$$\left| f(x) - f(a) - \int_a^x q(t) \prod_{i=1}^{k} (t - x_i)\, dt \right| \leqslant 2(b - a)^{k+1} E_{n-k-1}(g).$$

If we let

$$p(x) = f(a) + \int_a^x q(t) \prod_{i=1}^{k} (t - x_i)\, dt,$$

then $p \in P_n$, p is comonotone with f, and $\|f - p\| \leqslant 2(b-a)^{k+1} E_{n-k-1}(g)$. Since $g \in C^j[a, b]$ and $\|g^{(j)}\| \leqslant \|f^{(j+k+1)}\|$, there exists a_j such that

$$E_{n-k-1}(g) \leqslant \frac{a_j \|f^{(j+k+1)}\|}{(n-k)(n-k-1) \cdots (n-k-j+1)} \quad \text{for} \quad n > (k+j), \quad [1],$$

$$\leqslant \frac{2a_j \|f^{(j+k+1)}\|}{n^j} \quad \text{for} \quad n > 2(k+j).$$

Thus

$$\|f - p\| \leqslant \frac{4a_j(b-a)^{k+1} \|f^{(j+k+1)}\|}{n^j} = \frac{d_j(b-a)^{k+1} \|f^{(j+k+1)}\|}{n^j} \quad \text{for} \quad n > 2(k+j).$$

THEOREM 2. *Let f satisfy the hypotheses of Theorem* 1. *Then there exists $r_{j,k}$ such that, for $n > 4(k + j + 2)$,*

$$E_n^*(f) \leqslant \frac{(b-a)^{k+1} r_{j,k} \|f^{(j+k+1)}\|}{n^{j+k-1}},$$

where $r_{j,k}$ depends on $x_1, x_2, \ldots, x_k$ and j.

The proof of Theorem 2 is identical to that of Theorem 1, but makes use of parts (a) and (b) of Lemma 2 in the same way that Theorem 1 uses part (c) of that lemma.

Notice that the order of comonotone approximation in Theorem 2 is smaller than that in Theorem 1. On the other hand, the constant $r_{j,k}$ in Theorem 2 depends upon the location of the peaks of f, while the constant d_j in Theorem 1 is independent of f, n, and k.

REFERENCES

1. E. W. CHENEY, "Introduction to Approximation Theory," McGraw–Hill, New York, 1966, pp. 147–148.
2. E. ISAACSON AND H. B. KELLER, "Analysis of Numerical Methods," Wiley, New York, 1966.
3. G. G. LORENTZ AND K. L. ZELLER, Degree of approximation by monotone polynomials, I., *J. Approx. Theor.* **1** (1968), 501–504.
4. D. J. NEWMAN, E. PASSOW AND L. RAYMON, Piecewise monotone polynomial approximation. *Trans. Amer. Math. Soc.* **172** (1972), 465–472.
5. W. WOLIBNER, Sur un polynôme d'interpolation, *Colloq. Math.* **2** (1951), 136–137.